$\boxed{1}$ a) (I) $M+I$ inversible $\Longleftrightarrow |M+I| \neq 0$

$M^3+M+I=0 \Rightarrow M+I=-1\cdot M^3 \Rightarrow |M+I| = (-1)^3 |M^3| = (-1)^3 \cdot |M|^3 = -1\cdot(-2)^3 = 8 \neq 0$

Por tanto $M+I$ es inversible

(II) $2M+2I \Rightarrow |2M+2I| = |2(M+I)| = 2^3 \cdot |M+I| = 8\cdot 8 = 64$

b)

$$|B| = \begin{vmatrix} 3 & 8 & 4 \\ 1 & 8/3 & 4/3 \\ 2 & 0 & 4 \end{vmatrix} + \begin{vmatrix} x & y & z \\ 1 & 8/3 & 4/3 \\ 2 & 0 & 4 \end{vmatrix} = \frac{1}{3}\begin{vmatrix} 3 & 8 & 4 \\ 3 & 8 & 4 \\ 2 & 0 & 4 \end{vmatrix} + \frac{1}{3}\begin{vmatrix} x & y & z \\ 3 & 8 & 4 \\ 2 & 0 & 4 \end{vmatrix} =$$

$$= 0 + \frac{1}{3}\cdot 2 \begin{vmatrix} x & y & z \\ 3 & 8 & 4 \\ 1 & 0 & 2 \end{vmatrix} = -\frac{2}{3}\begin{vmatrix} x & y & z \\ 1 & 0 & 2 \\ 3 & 8 & 4 \end{vmatrix} = -\frac{2}{3}\cdot 1 = -\frac{2}{3}$$

$\boxed{2}$
$$\begin{cases} x + y + mz = m \\ mx + (m-1)y + z = 2 \\ x + y + z = 1 \end{cases}$$

a) Sea $A = \begin{pmatrix} 1 & 1 & m \\ m & m-1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ la matriz del sistema, y $A^* = \begin{pmatrix} 1 & 1 & m & m \\ m & m-1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ la matriz ampliada.

1º) Estudiamos el rango de A en función de los valores de m

$$|A| = \begin{vmatrix} 1 & 1 & m \\ m & m-1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & m-1 \\ m & -1 & 1-m \\ 1 & 0 & 0 \end{vmatrix} = -1\cdot(-1)^{2+2}\begin{vmatrix} 1 & m-1 \\ 1 & 0 \end{vmatrix} = m-1$$

$C_3 - C_1 \to C_3$
$C_2 - C_1 \to C_2$

- $m \neq 1 \Rightarrow$ rango(A) = rango(A^*) = nº incógnitas. Por el Teorema de Rouché-Fröbenius, el sistema es compatible determinado (el sistema tiene una única solución
- $m=1$:

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ Como $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0 \Rightarrow$ rango$(A) = 2$

$A^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ rango(A^*) = rango$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} = 2$

$F_1 = F_3$

rango(A) = rango(A^*) = $2 <$ nº incógnitas. $\Rightarrow$ Por el Teorema de Rouché-Fröbenius, el sistema es compatible indeterminado Es decir, tiene infinitas soluciones.

b) $m=1$:

$$\begin{cases} x+y+z=1 \\ x \quad +z=2 \\ x+y+z=1 \end{cases} \Longrightarrow \begin{array}{l} x=2-z \\ y=1-x-z=-1 \end{array} \Longrightarrow \text{Solución } (2-\lambda, -1, \lambda)$$

c) $C=(1 \ -1 \ 0) \qquad D=(1 \ 2 \ -1)$

$$C^t \cdot D = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}(1 \ 2 \ -1) = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \operatorname{rango}(C^t D)=1$$

$F_3=0$

$F_1=-F_2$

$\boxed{3}$ $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B=\begin{pmatrix} x & y \\ z & 0 \end{pmatrix}$

a) $A \cdot B = B \cdot A \Longleftrightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ z & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Longleftrightarrow \begin{cases} x+2z = x+3y \\ y = 2x+4y \\ 3x+4z = z \\ 3y = 2z \end{cases} \Longleftrightarrow \begin{cases} 3y=2z \\ -3y=2x \\ 3x=-3z \end{cases} \Longrightarrow (x,y,z)=\left(-\lambda, \frac{2\lambda}{3}, \lambda\right)$

$B=\begin{pmatrix} -\lambda & \frac{2}{3}\lambda \\ \lambda & 0 \end{pmatrix}$

b) B inversible $\Longleftrightarrow |B| \neq 0 \Longleftrightarrow -\frac{2}{3}\lambda^2 \neq 0 \Longleftrightarrow \lambda \neq 0$

$B^{-1} = \frac{1}{|B|}(\operatorname{Adj}(B))^t \qquad (\operatorname{Adj}(B))^t = \begin{pmatrix} 0 & -\frac{2\lambda}{3} \\ -\lambda & -\lambda \end{pmatrix} \Longrightarrow B^{-1} = \begin{pmatrix} 0 & 1/\lambda \\ 3/2\lambda & 3/2\lambda \end{pmatrix}$

$\boxed{4}$ $A=\begin{pmatrix} 1 & a+1 & 2 \\ a & 1 & 1 \\ 1 & -1 & a \end{pmatrix}$

a) $|A| = a-2a+a+1-2-a^2(a+1)+1 = -a^2(a+1)$

b) A inversible $\Longleftrightarrow |A| \neq 0 \Longleftrightarrow a \neq 0$ y $a \neq -1$

c) Si $a=0$, A no es inversible

$A=\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad AX=0 \Longrightarrow \begin{cases} x+y+2z=0 \\ y+z=0 \\ x-y=0 \end{cases} \Longrightarrow \begin{cases} y+z=0 \\ x-y=0 \end{cases} \Longrightarrow \begin{array}{l} x=y \\ z=-y \end{array}$

En A, $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \neq 0 \Rightarrow \operatorname{rango} A = 2$

Solución $(\lambda, \lambda, -\lambda)$

$\boxed{5}$ $A=\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & m^2 \\ 1 & -1 & m^2-m \end{pmatrix}$

a) $|A| \underset{C_1+C_2 \to C_1}{=} \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & m^2 \\ 0 & -1 & m^2-m \end{vmatrix} = 1 \cdot (-1)^{2+1} \cdot \begin{vmatrix} -1 & 0 \\ -1 & m^2-m \end{vmatrix} = m^2-m = m(m-1)$

$\text{rango}(A)=3 \iff m \neq 0$ y $m \neq 1$

· $m=0 \implies A=\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} \quad \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \neq 0 \implies \text{rango}(A)=2$

· $m=1 \implies A=\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \implies \text{rango}(A)=2$

b) $m=2 \implies A$ inversible $(\text{rango}(A)=3)$

$3XA - A = I \implies 3XA = A+I \implies 3X = (A+I)A^{-1} \implies 3X = I + A^{-1}$

$\implies X = \frac{1}{3}(I+A^{-1})$

$A=\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 2 \end{pmatrix} \qquad \text{Adj}(A)=\begin{pmatrix} 4 & 2 & -1 \\ 2 & 2 & 0 \\ -4 & -4 & 1 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|}\left[\text{Adj}(A)\right]^t = \frac{1}{2}\begin{pmatrix} 4 & 2 & -4 \\ 2 & 2 & -4 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 1 & -2 \\ -1/2 & 0 & 1/2 \end{pmatrix}$

$X = \frac{1}{3}(I+A^{-1}) = \frac{1}{3}\begin{pmatrix} 3 & 1 & -2 \\ 1 & 2 & -2 \\ -1/2 & 0 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 1/3 & -2/3 \\ 1/3 & 2/3 & -2/3 \\ -1/6 & 0 & 1/2 \end{pmatrix}$

$\boxed{6}$ $A=\begin{pmatrix} t & 2 \\ 1 & 1 \end{pmatrix}$ $B=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

a) $A^2X+AX=B \Rightarrow (A^2+A)X=B \Rightarrow A(A+I)X=B$

Necesitamos que $A(A+I)$ sea inversible:

$A(A+I)$ inversible $\Leftrightarrow |A(A+I)|\neq 0 \Leftrightarrow |A|\neq 0$ y $|A+I|\neq 0$

$|A\cdot(A+I)|=|A|\cdot|A+I|$

$|A|=t-2$ $|I+A|=\begin{vmatrix} t+1 & 2 \\ 1 & 2 \end{vmatrix}=2t+2-2=2t$

Por tanto $|A(I+A)|=(t-2)2t\neq 0 \Leftrightarrow t\neq 2$ y $t\neq 0$

b) Si $t=1$ $A=\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$A(A+I)X=B \Longrightarrow X=\left[A(A+I)\right]^{-1}\cdot B$

$A\cdot(A+I)=\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}=\begin{pmatrix} 4 & 6 \\ 3 & 4 \end{pmatrix}$ $|A(A+I)|=-2$

$\left[A(A+I)\right]^{-1}=-\frac{1}{2}\begin{pmatrix} 4 & -6 \\ -3 & 4 \end{pmatrix}=\begin{pmatrix} -2 & 3 \\ 3/2 & -2 \end{pmatrix}$

$X=\begin{pmatrix} -2 & 3 \\ 3/2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}=\begin{pmatrix} -2 & -3 \\ 3/2 & 2 \end{pmatrix}$

c) $t=-1 \Longrightarrow A=\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \Longrightarrow A^2=\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}=3I$

$A^4=(A^2)^2=(3I)^2=9I=\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$

$A^{16}=(A^2)^8=(3I)^8=3^8 I=\begin{pmatrix} 6561 & 0 \\ 0 & 6561 \end{pmatrix}$