$\boxed{21}$ a) $\displaystyle\int_0^{\pi/4} \frac{\operatorname{sen} x}{3+\operatorname{sen}^2 x}\,dx \underset{\operatorname{sen}^2 x = 1-\cos^2 x}{=} \int_0^{\pi/4} \frac{\operatorname{sen} x}{4-\cos^2 x}\,dx$

$$\int \frac{\operatorname{sen} x}{4-\cos^2 x}\,dx \underset{\substack{\cos x = t \\ -\operatorname{sen} x\, dx = dt}}{=} -\int \frac{dt}{4-t^2} \underset{*}{=} -\int \frac{1/4}{2-t}\,dt - \int \frac{1/4}{2+t}\,dt =$$

$$* \quad \frac{1}{4-t^2} = \frac{A}{2-t} + \frac{B}{2+t}$$
$$1 = A(2+t) + B(2-t)$$
$$t=-2 \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$$
$$t = 2 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$$

$$= \frac{1}{4}\ln|2-t| - \frac{1}{4}\ln|2+t| = \frac{1}{4}\ln(2-\cos x) - \frac{1}{4}\ln(2+\cos x)$$
$$= \frac{1}{4}\ln\left(\frac{2-\cos x}{2+\cos x}\right) + C$$

Por tanto

$$\int_0^{\pi/4} \frac{\operatorname{sen} x}{3+\operatorname{sen}^2 x}\,dx = \left[\frac{1}{4}\ln\left(\frac{2-\cos x}{2+\cos x}\right)\right]_0^{\pi/4} = \frac{1}{4}\ln\left(\frac{4-\sqrt{2}}{4+\sqrt{2}}\right) - \frac{1}{4}\ln\left(\frac{1}{3}\right)$$

$$\frac{2-\frac{\sqrt{2}}{2}}{2+\frac{\sqrt{2}}{2}} = \frac{4-\sqrt{2}}{4+\sqrt{2}} = \frac{(4-\sqrt{2})^2}{14}$$

b) $\displaystyle\int_0^{1/2} \frac{x^3}{\sqrt{1-x^2}}\,dx$

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx \underset{\substack{x = \operatorname{sen} t \\ dx = \cos t\, dt}}{=} \int \frac{\operatorname{sen}^3 t \cos t\, dt}{\sqrt{1-\operatorname{sen}^2 t}} = \int \frac{\operatorname{sen}^3 t \cos t}{\cos t}\,dt =$$

$$= \int \operatorname{sen}^3 t\, dt = \int \operatorname{sen}^2 t \operatorname{sen} t\, dt = \int (1-\cos^2 t)\operatorname{sen} t\, dt =$$

$$= \int \operatorname{sen} t - \cos^2 t \operatorname{sen} t\, dt = -\cos t + \frac{\cos^3 t}{3} = -\cos t + \frac{\cos t(1-\operatorname{sen}^2 t)}{3} =$$

$$= -\cos t + \frac{1}{3}\cos t - \frac{1}{3}\cos t \operatorname{sen}^2 t = -\frac{2}{3}\cos t - \frac{1}{3}\cos t \operatorname{sen}^2 t =$$

$$= -\frac{2}{3}\sqrt{1-\operatorname{sen}^2 t} - \frac{1}{3}\sqrt{1-\operatorname{sen}^2 t}\cdot \operatorname{sen}^2 t \underset{x = \operatorname{sen} t}{=} -\frac{2}{3}\sqrt{1-x^2} - \frac{1}{3}x^2\sqrt{1-x^2} + C$$

Por tanto $\displaystyle\int_0^{1/2} \frac{x^3}{\sqrt{1-x^2}}\,dx = \left[-\frac{2}{3}\sqrt{1-x^2} - \frac{1}{3}x^2\sqrt{1-x^2}\right]_0^{1/2} =$

$$= -\frac{2}{3}\cdot\frac{\sqrt{3}}{2} - \frac{1}{3}\cdot\frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{2}{3} = \frac{16-9\sqrt{3}}{24}$$

$\boxed{21}$ c) $\displaystyle\int_0^{\pi^2/4} \frac{\cos\sqrt{x}}{2}\,dx$

$$\int \frac{\cos\sqrt{x}}{2}\,dx = \int \frac{(\cos t)\cdot 2t}{2}\,dt = \int t\cos t\,dt = t\,\mathrm{sen}\,t - \int \mathrm{sen}\,t\,dt =$$

$t^2 = x \rightarrow 2t\,dt = dx$

$u = t \rightarrow du = dt$

$dv = \cos t\,dt \rightarrow v = \mathrm{sen}\,t$

$$= t\,\mathrm{sen}\,t + \cos t = \sqrt{x}\,\mathrm{sen}\sqrt{x} + \cos\sqrt{x} + C$$

$$\int_0^{\pi^2/4} \frac{\cos\sqrt{x}}{2}\,dx = \left[\sqrt{x}\,\mathrm{sen}\sqrt{x} + \cos\sqrt{x}\right]_0^{\pi^2/4} = \frac{\pi}{2} - 1$$

d) $\displaystyle\int_1^3 \frac{x^4-1}{x^4+x^2}\,dx$

$$\int \frac{x^4-1}{x^4+x^2}\,dx = \int \frac{x^4+x^2-x^2-1}{x^4+x^2}\,dx = \int 1 - \frac{x^2+1}{x^2(x^2+1)}\,dx =$$

$$= \int 1 - \frac{1}{x^2}\,dx = x + \frac{1}{x} + C$$

$$\int_1^3 \frac{x^4-1}{x^4+x^2}\,dx = \left[x + \frac{1}{x}\right]_1^3 = 3 + \frac{1}{3} - (1+1) = \frac{4}{3}$$

f) $$\int_0^{\pi/2} \cos^3(x)\,dx = \int_0^{\pi/2} \cos^2(x)\cos(x)\,dx = \int_0^{\pi/2} (1-\mathrm{sen}^2 x)\cos x\,dx =$$

$$= \int_0^{\pi/2} \cos x - \mathrm{sen}^2 x\cos x\,dx = \left[\mathrm{sen}\,x - \frac{\mathrm{sen}^3 x}{3}\right]_0^{\pi/2} = \frac{2}{3}$$

$\boxed{21}$ a) $\displaystyle\int_1^6 \frac{1}{x} - \ln x \, dx = \int_1^6 \frac{1}{x} dx - \int_1^6 \ln x \, dx$

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x\ln x - \int 1 \, dx = x \ln x - x$$

$u = \ln x \longrightarrow du = \frac{1}{x} dx$

$dv = dx \longrightarrow v = x$

$$\int_1^6 \frac{1}{x} - \ln x \, dx = \Big[\ln x - x\ln x + x \Big]_1^6 = \ln 6 - 6\ln 6 + 6 - 1 =$$

$= 5 - 5\ln 6$

b) $\displaystyle\int_0^1 x \ln(1+x^2) \, dx$

$$\int x \ln(1+x^2) dx = \frac{x^2}{2} \ln(1+x^2) - \int \frac{x^3}{1+x^2} dx$$

$u = \ln(1+x^2) \longrightarrow du = \frac{2x}{1+x^2} dx$

$dv = x\,dx \longrightarrow v = \frac{x^2}{2}$

$$\begin{array}{l} x^3 \\ \underline{-x^3 - x} \\ \quad / \; -x \end{array} \quad \begin{array}{|l} x^2+1 \\ \hline x \end{array}$$

$$\int \frac{x^3}{1+x^2} dx = \int x - \frac{x}{1+x^2} \, dx = \int x - \frac{1}{2} \cdot \frac{2x}{1+x^2} dx =$$

$= \frac{x^2}{2} - \frac{1}{2} \ln(1+x^2)$

$$\int_0^1 x \ln(1+x^2) dx = \left[\frac{x^2}{2} \ln(1+x^2) - \frac{x^2}{2} + \frac{1}{2} \ln(1+x^2) \right]_0^1 =$$

$= \ln 2 - \frac{1}{2}$

$$\boxed{21}\ \text{i)}\ \int_2^3 \frac{5x^3-3x+1}{x^3-x}\,dx = \int_2^3 \frac{5x^3-5x+2x+1}{x^3-x}\,dx = \int_2^3 5+\frac{2x+1}{x^3-x}\,dx$$

$$\int \frac{2x+1}{x^3-x}\,dx = \int\frac{dx}{x} - \frac{3}{2}\int\frac{dx}{x+1} + \frac{1}{2}\int\frac{dx}{x-1} = \ln|x| - \frac{3}{2}\ln|x+1| + \frac{1}{2}\ln|x-1|$$

$$\frac{2x+1}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$2x+1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

$$x=1 \Rightarrow 3 = 2C \Rightarrow C = \frac{3}{2}$$

$$x=0 \Rightarrow 1 = -A \Rightarrow A = -1$$

$$x=-1 \Rightarrow -1 = 2B \Rightarrow B = -\frac{1}{2}$$

$$\int_2^3 5 + \frac{2x-1}{x^3-x}\,dx = \left[5x - \ln|x| - \frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x-1|\right]_2^3 =$$

$$15 - \ln 3 - \frac{1}{2}\ln \overset{2^2}{4} + \frac{3}{2}\ln 2 - \left(10 - \ln 2 - \frac{1}{2}\ln 3\right) =$$

$$= 15 - \ln 3 - \ln 2 + \frac{3}{2}\ln 2 - 10 + \ln 2 + \frac{1}{2}\ln 3 =$$

$$= 5 - \frac{1}{2}\ln 3 + \frac{3}{2}\ln 2$$

$$\text{j)}\ \int_1^e \frac{(x-1)^2}{x^2+1}\,dx = \int_1^e \frac{x^2-2x+1}{x^2+1}\,dx = \int_1^e 1 - \frac{2x}{1+x^2}\,dx =$$

$$= \left[x - \ln(1+x^2)\right]_1^e = e - \ln(1+e^2) - 1 + \ln 2 = e - 1 + \ln\left(\frac{2}{1+e^2}\right)$$