Sequences

A list of numbers in a particular order, that follow some rule for finding later values, is called a **sequence**. Each number in a sequence is called a **term**, and terms are often denoted by $a_1, a_2, a_3, \ldots, a_n, \ldots$.

- One way to define a sequence is to give a **formula for the nth term**

Example: the sequence 1, 4, 9, 16,... is produced by the formula for the nth term $a_n = n^2$

$a_{13} = 13^2 = 169$
$a_{20} = 20^2 = 400$
etc.
} some others

- Another way to define a sequence is to give a **starting value** together with a rule that shows the connection between successive terms. This is sometimes called a **recursive definition**. (Or recurrence relation)

Example: the sequence 5, 7, 12, 19, 31,... is defined by the recurrence relation below

$a_1 = 5 \qquad a_2 = 7 \qquad a_n = a_{n-2} + a_{n-1}$

$a_6 = a_4 + a_5 = 19 + 31 = 50$
$a_7 = a_5 + a_6 = 31 + 50 = 81$
} some other terms

Progressions

Two special sequences are the **arithmetic progression** (A.P.) and the **geometric progression** (G.P.).

- In an A.P. successive terms have a **common difference**, e.g. 1, 4, 7, 10, ...
  The terms of an A.P. take the form $a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots$ and the nth term is given by $a_n = a_1 + (n-1)d$.

Example: the arithmetic progression with first term 6 and difference 4 (6, 10, 14, 18, 22,...) is defined by $a_n = 6 + 4 \cdot (n-1)$

- In a G.P. successive terms are connected by a **common ratio**, e.g. 3, 6, 12, 24, ...
  The terms of a G.P. take the form $q, qr, qr^2, qr^3, \ldots$ and the nth term is given by $a_n = qr^{n-1}$.

Example: the geometric progression with first term 3 and common ratio 2 (3, 6, 12, 24, 48,...) is defined by $a_n = 3 \cdot 2^{n-1}$

Series

A **series** is formed by adding together the terms of a sequence.

The sum of the first n terms of a series is often denoted by S_n and so $S_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{i=1} u_i$

The sum of the first n terms of an arithmetic series is: $S_n = \frac{n}{2}(a_1 + a_n)$.

Substituting $a_n = a_1 + (n-1)d$ gives the alternative form of the result:

$$S_n = \frac{n}{2}(2a_1 + (n-1)d).$$

For example, the sum of the first 100 natural numbers is $\frac{100}{2}(100 + 1) = 5050$.

Example Find the sum of the first 50 terms of the series $15 + 18 + 21 + 24 + \ldots$

In this series, $a = 15$, $d = 3$ and $n = 50$.

Using $S_n = \frac{n}{2}(2a + (n-1)d)$ gives $S_{50} = \frac{50}{2}(2 \times 15 + 49 \times 3) = 4425$.

The sum of the first n terms of a geometric series is $S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1(r^n - 1)}{r - 1}$

Example Find the sum of the first 20 terms of the series $8 + 12 + 18 + 27 + \ldots$ to the nearest whole number.

In this series, $a = 8$, $r = 1.5$ and $n = 20$.

This gives $S_{20} = \frac{8(1.5^{20} - 1)}{1.5 - 1} = 53188.107\ldots = 53188$ to the nearest whole number.

Note Provided that $|r| < 1$, the sum of a geometric series converges to $\frac{a}{1-r}$ as n tends to infinity. This is known as the **sum to infinity** of a geometric series.

For example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1 - \frac{1}{2}} = 2$

EXERCISES

1) Write down the first five terms of the sequence given by:
(a) $u_n = 2n^2$ (b) $u_1 = 10$, $u_{n+1} = 3u_n + 2$.

2) Find the 20th term of an A.P. with first term 7 and common difference 5.

3) Find the sum of the first 1000 natural numbers.

4) Find the sum of the first 20 terms of $12 + 15 + 18.75 + \ldots$ to 4 s.f.

5) Explain why the series $10 + 9 + 8.1 + 7.29 + \ldots$ is convergent and find the value of its sum to infinity to 5 s.f.

6) Write down the first six terms of the following sequences:
a) Each term is calculated adding 3 to the previous term. The first term is –8.
b) The first term is 16. The following terms are obtained by multiplying the previous one by 0.5.
c) The first term is 36, the second term is 12 and the following terms are half the addition of the two previous terms.
d) The first one is 2. Each term is the inverse of the previous one.

7) Write down the 10th, the 25th and the 100th terms of the sequences given by:

$a_n = 2n-3$ $\qquad a_n = \dfrac{n+1}{2}$ $\qquad a_n = 1-n^2$ $\qquad a_n = 1+\dfrac{(-1)^n}{n}$

8) Write down the first five terms of the sequences given by the formulae for the nth term: $a_n = 10-5n$ $\qquad b_n = \dfrac{n^2-1}{n}$ $\qquad c_n = 3^{n-2}$ $\qquad d_n = \dfrac{2n-1}{n+1}$
One of them is an arithmetic progression and another one is a geometric progression. Which ones?

9) Find the common difference, write down the formula for the nth term and calculate the sum of the first 20 terms of the arithmetic progressions below:
a) 1, 1.5, 2, 2.5,… b) 5, 3, 1, –1,… c) 3.3, 4.4, 5.5, 6.6,... d) $\dfrac{1}{4}, \dfrac{5}{4}, \dfrac{9}{4}, \dfrac{13}{4}, \ldots$

10) Find the common ratio, write down the formula for the nth term and calculate the sum of the first 10 terms of the geometric progressions below:
a) 0.25, 0.75, 2.25, 6.75,… b) 3, –6, 12, –24,… c) 4, 6, 9, 13.5,…

11) Find the formula for the nth term and the sum of the first 15 terms of the following sequences: a) 8, 5, 2, –1,… b) $\dfrac{1}{81}, \dfrac{1}{27}, \dfrac{1}{9}, \dfrac{1}{3}, \ldots$

12) Spot the arithmetic progressions, the geometric progressions and the sequences that are not these types. Find the formula for the nth term in all of them.
a) 1, 1, 1, 1,… b) $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \ldots$ c) $\dfrac{7}{8}, 1, \dfrac{9}{8}, \dfrac{5}{4}, \ldots$
d) $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$ e) $\sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$ f) $1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \ldots$

13) Write down the formula for the n^th term of an arithmetic progression such that $a_1 = 7$ and $a_4 = 40$

14) An arithmetic progression has term $a_8 = 4$ and the common difference is $d = -5$. Calculate the first term and the sum of the first 25 terms.

15) A geometric progression has first term $a_1 = 64$ and common ratio is $r = 0.25$. Calculate the first term that is not an integer number. Write down a_{25} without approximating.

16) Calculate the sum of the first five terms of a geometric progression such that $a_1 = 1000$ and $a_4 = 8$.

17) The angles of a triangle form an arithmetic progression being 36° the measure of the smaller one. Find out the measure of the other angles.

18) A person goes on holiday and spends 100€ on the first day, 95€ on the second day and so on (each day 5€ less than the previous day are spent). The money will last for 20 days. What amount of money has that person got to spend during the holiday?

19) Find the sum of the first twelve terms of an arithmetic progression such that $a_3 = 24$ and $a_{10} = 66$.

20) A species of bacteria reproduces by bipartition every 10 minutes. How many bacteria will there be after 8 hours?