STRAND F: ALGEBRA

Unit F4 *Solving Quadratic Equations*

Text

Contents

F4 Solving Quadratic Equations

F4.1 Factorisation

Equations of the form

$$ax^2 + bx + c = 0$$

are called *quadratic* equations. Many can be solved using factorisation. If a quadratic equation can be written as

$$(x - a)(x - b) = 0$$

then the equation will be satisfied if *either* bracket is equal to zero. That is,

$$(x - a) = 0 \quad \text{or} \quad (x - b) = 0$$

So there would be two possible solutions, $x = a$ and $x = b$.

Worked Example 1

Solve $x^2 + 6x + 5 = 0$.

Solution

Factorising gives

$$(x + 5)(x + 1) = 0$$

So

$$x + 5 = 0 \quad \text{or} \quad x + 1 = 0$$

therefore

$$x = -5 \quad \text{or} \quad x = -1$$

Worked Example 2

Solve $x^2 + 5x - 14 = 0$.

Solution

Factorising gives

$$(x - 2)(x + 7) = 0$$

So

$$x - 2 = 0 \quad \text{or} \quad x + 7 = 0$$

therefore

$$x = 2 \quad \text{or} \quad x = -7$$

Worked Example 3

Solve $x^2 - 12x = 0$.

Solution

Factorising gives

$$x(x - 12) = 0$$

So

$$x = 0 \quad \text{or} \quad x - 12 = 0$$

therefore

$$x = 0 \quad \text{or} \quad x = 12$$

Worked Example 4

Solve

$$4x^2 - 81 = 0$$

Solution

Factorising gives

$$(2x - 9)(2x + 9) = 0$$

So

$$2x - 9 = 0 \quad \text{or} \quad 2x + 9 = 0$$

therefore

$$x = \frac{9}{2} \quad \text{or} \quad x = -\frac{9}{2}$$

$$= 4\frac{1}{2} \qquad\qquad = -4\frac{1}{2}$$

Worked Example 5

Solve $x^2 - 4x + 4 = 0$.

Solution

Factorising gives

$$(x - 2)(x - 2) = 0$$

So

$$x - 2 = 0 \quad \text{or} \quad x - 2 = 0$$

therefore

$$x = 2 \quad \text{or} \quad x = 2$$

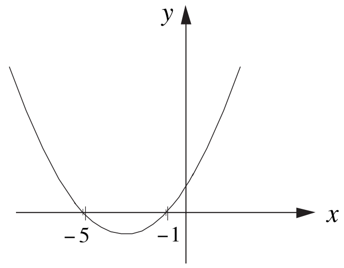
This type of solution is often called a *repeated* solution and results from solving a perfect square, that is

$$(x - 2)^2 = 0$$

Most of these examples have had two solutions, but the last example had only one solution.

The graphs below show

$y = x^2 + 6x + 5$ and $y = x^2 - 4x + 4$.

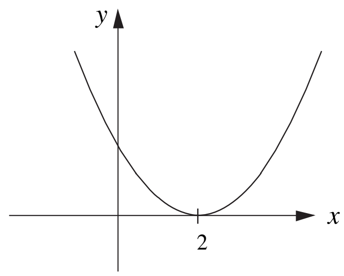


The curve crosses the x-axis at

$x = -5$ and $x = -1$.

These are the solutions of

$x^2 + 6x + 5 = 0$



The curve touches the x-axis at

$x = 2$

This is the solution of

$x^2 - 4x + 4 = 0$

Exercises

1. Solve the following quadratic equations.

(a) $x^2 + x - 12 = 0$	(b) $x^2 - 2x - 15 = 0$	(c) $x^2 + 4x - 12 = 0$
(d) $x^2 + 6x = 0$	(e) $3x^2 - 4x = 0$	(f) $4x^2 - 9x = 0$
(g) $x^2 - 9 = 0$	(h) $x^2 - 49 = 0$	(i) $9x^2 - 64 = 0$
(j) $x^2 - 8x + 16 = 0$	(k) $x^2 + 10x + 25 = 0$	(l) $x^2 - 3x - 18 = 0$
(m) $x^2 - 11x + 28 = 0$	(n) $x^2 + x - 30 = 0$	(o) $x^2 - 14x + 40 = 0$
(p) $2x^2 + 7x + 3 = 0$	(q) $2x^2 + 5x - 12 = 0$	(r) $3x^2 - 7x + 4 = 0$
(s) $4x^2 + x - 3 = 0$	(t) $2x^2 + 5x - 3 = 0$	(u) $2x^2 - 19x + 35 = 0$

2. The equations of a number of curves are given below. Find where each curve crosses the x-axis and use this to draw a sketch of the curve.

(a) $y = x^2 + 6x + 9$ (b) $y = x^2 - 4$

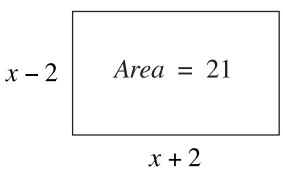
(c) $y = 2x^2 - 3x$ (d) $y = x^2 + x - 12$

3. Use the difference of two squares result to solve the following equations.

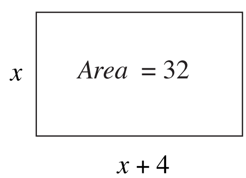
(a) $x^4 - 16 = 0$ (b) $x^4 - 625 = 0$

4. Find the lengths of each side of the following rectangles.

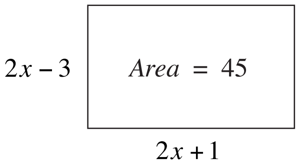
(a)



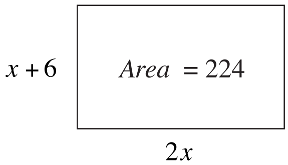
(b)



(c)



(d)

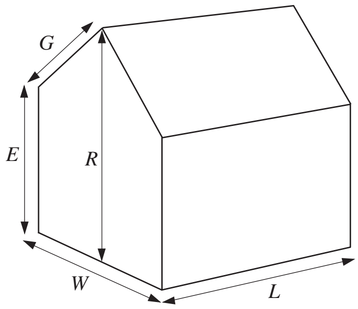


5. The height of a ball thrown straight up from the ground into the air at time, t, is given by

$$h = 8t - 10t^2$$

Find the time it takes for the ball to go up and fall back to ground level.

6. The diagram represents a shed.



The volume of the shed is given by the formula

$$V = \frac{1}{2} LW(E + R)$$

(a) Make L the subject of the formula, giving your answer as simply as possible.

The surface area, A, of the shed, is given by the formula

$$A = 2GL + 2EL + W(E + R)$$

where $V = 500$, $A = 300$, $E = 6$ and $G = 4$.

(b) By substituting these values into the equations for V and A show that L satisfies the equation

$$L^2 - 15L + 50 = 0$$

Make the steps in your working clear.

(c) Solve the equation $L^2 - 15L + 50 = 0$.

F4.2 Using the Formula

The formula given below is particularly useful for quadratics which cannot be factorised. To prove this important result requires some quite complex analysis, using a technique called *completing the square,* which is the subject of Section F4.3.

Theorem

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof

The equation $ax^2 + bx + c = 0$ is first divided by the non-zero constant, a, giving

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Note that

$$\begin{aligned}\left(x + \frac{b}{2a}\right)^2 &= \left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right)\\ &= x^2 + \frac{bx}{2a} + \frac{bx}{2a} + \left(\frac{b}{2a}\right)^2 \quad \text{(expanding)}\\ &= x^2 + \frac{2bx}{2a} + \left(\frac{b}{2a}\right)^2 \quad \text{(adding like terms)}\\ &= x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 \quad \text{(simplifying)}\end{aligned}$$

The first two terms are identical to the first two terms in our equation, so you can re-write the equation as

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\begin{aligned}\left(x + \frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a}\\ &= \frac{b^2}{4a^2} - \frac{c}{a}\end{aligned}$$

i.e. $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking the square root of both sides of the equation gives

$$\begin{aligned}x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}\\ &= \frac{\pm\sqrt{b^2 - 4ac}}{2a}\end{aligned}$$

Hence

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

or $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

as required.

Worked Example 1

Solve

$$x^2 + 6x - 8 = 0$$

giving the solution correct to 2 decimal places.

Solution

Here $a = 1$, $b = 6$ and $c = -8$. These values can be substituted into

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to give

$$x = \frac{-6 \pm \sqrt{6^2 - (4 \times 1 \times -8)}}{2 \times 1}$$

$$= \frac{-6 \pm \sqrt{68}}{2}$$

$$= \frac{-6 + \sqrt{68}}{2} \quad \text{or} \quad \frac{-6 - \sqrt{68}}{2}$$

$$= 1.12 \quad \text{or} \quad -7.12 \quad \text{(to 2 d.p.)}$$

Worked Example 2

Solve the quadratic equation

$$4x^2 - 12x + 9 = 0.$$

Solution

Here $a = 4$, $b = -12$ and $c = 9$. Substituting the values into

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives

$$x = \frac{12 \pm \sqrt{(-12)^2 - (4 \times 4 \times 9)}}{2 \times 4}$$

$$= \frac{12 \pm \sqrt{144 - 144}}{8}$$

$$= \frac{12 \pm \sqrt{0}}{8}$$

$$= \frac{12}{8} \quad \left(= \frac{3}{2}\right)$$

$$= 1.5$$

Worked Example 3

Solve the quadratic equation

$$x^2 + x + 5 = 0$$

Solution

Here $a = 1$, $b = 1$ and $c = 5$. Substituting the values into the formula gives

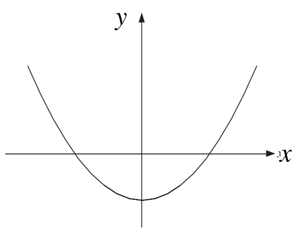
$$x = \frac{-1 \pm \sqrt{1^2 - (4 \times 1 \times 5)}}{2 \times 1}$$

$$= \frac{-1 \pm \sqrt{1 - 20}}{2}$$

$$= \frac{-1 \pm \sqrt{-19}}{2}$$

As it is not possible to find $\sqrt{-19}$, this equation has *no* solutions.

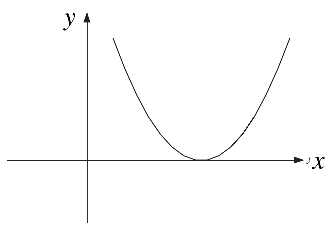
These three examples illustrate that a quadratic equation can have 2, 1 or 0 solutions. The graphs below illustrate these graphically and show how the number of solutions depends on the sign of $\left(b^2 - 4ac\right)$ which is part of the quadratic formula.



Two solutions

$b^2 - 4ac > 0$

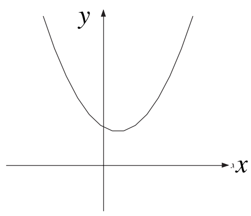
(Worked Example 1)



One solution

$b^2 - 4ac = 0$

(Worked Example 2)



No solutions

$b^2 - 4ac < 0$

(Worked Example 3)

Exercises

1. Use the quadratic equation formula to find the solutions, where they exist, of each of the following equations. Give answers to 2 decimal places.

(a)	$4x^2 - 7x + 3 = 0$	(b)	$2x^2 + x - 10 = 0$	(c)	$9x^2 - 6x - 11 = 0$
(d)	$3x^2 - 5x - 7 = 0$	(e)	$x^2 + x - 8 = 0$	(f)	$4x^2 - 6x - 9 = 0$
(g)	$2x^2 + 17x - 9 = 0$	(h)	$x^2 - 14x = 0$	(i)	$x^2 + 2x - 10 = 0$
(j)	$3x^2 + 8x - 1 = 0$	(k)	$x^2 + 6 = 0$	(l)	$2x^2 - 8x + 3 = 0$
(m)	$4x^2 - 5x - 3 = 0$	(n)	$5x^2 - 4x + 12 = 0$	(o)	$x^2 - 6x - 5 = 0$

2. A ticket printing and cutting machine cuts rectangular cards which are 2 cm longer than they are wide.

   (a) If x is the width of a ticket, find an expression for the area of the ticket.

   (b) Find the size of a ticket with an area of 10 cm^2.

3. A window manufacturer makes a range of windows for which the height is 0.5 m greater than the width.

   Find the width and height of a window with an area of 2 m^2.

4. The height of a stone launched from a catapult is given by

$$h = 20t - 9.8t^2$$

   where t is the time after the moment of launching.

   (a) Find when the stone hits the ground.

   (b) For how long is the stone more than 5 m above the ground?

   (c) Is the stone ever more than 12 m above ground level?

   (d) If m is the maximum height of the stone, write down a quadratic equation which involves m. Explain why this equation has only one solution and use this fact to find the value of m, to 2 decimal places.

5. The equation below is used to find the maximum amount, x, which a bungee cord stretches during a bungee jump:

$$mgx + mgl - \frac{1}{2}kx^2 = 0,$$

   where m = mass of bungee jumper
   l = length of rope when not stretched (10 m)
   k = stiffness constant $\left(120 \text{ N m}^{-1}\right)$
   g = acceleration due to gravity $\left(10 \text{ m s}^{-2}\right)$

   (a) Find the maximum amount that the cord stretches for a bungee jumper of mass 60 kg.

   (b) How much more would the cord stretch for a person of mass 70 kg?

6. Solve the equation $x^2 = 5x + 7$, giving your answers correct to 3 significant figures.

F4.3 Completing the Square

Completing the square is a technique which can be used to solve quadratic equations that do not factorise. It can also be useful when finding the minimum or maximum value of a quadratic.

A general quadratic $ax^2 + bx + c$ is written in the form $a(x+p)^2 + q$ when completing the square. You need to find the constants p and q so that the two expressions are identical.

Worked Example 1

Complete the square for $x^2 + 10x + 2$.

Solution

First consider the $x^2 + 10x$. These terms can be obtained by expanding $(x+5)^2$.

But $$(x+5)^2 = x^2 + 10x + 25$$

so $$x^2 + 10x = (x+5)^2 - 25$$

Therefore $$\begin{aligned} x^2 + 10x + 2 &= (x+5)^2 - 25 + 2 \\ &= (x+5)^2 - 23 \end{aligned}$$

Worked Example 2

Complete the square for $x^2 + 6x - 8$.

Solution

To obtain $x^2 + 6x$ requires expanding $(x+3)^2$.

But $$(x+3)^2 = x^2 + 6x + 9$$

so $$x^2 + 6x = (x+3)^2 - 9$$

Therefore $$\begin{aligned} x^2 + 6x - 8 &= (x+3)^2 - 9 - 8 \\ &= (x+3)^2 - 17 \end{aligned}$$

Note

When completing the square for $x^2 + bx + c$, the result is

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$$

and for $a \neq 0$,

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

Worked Example 3

Complete the square for $3x^2 + 6x + 7$.

Solution

As a first step, the quadratic can be rearranged as shown below.

$$3x^2 + 6x + 7 = 3\left(x^2 + 2x\right) + 7$$

Then note that $$x^2 + 2x = \left(x + 1\right)^2 - 1$$

so
$$\begin{aligned} 3\left(x^2 + 2x\right) + 7 &= 3\left[\left(x + 1\right)^2 - 1\right] + 7 \\ &= 3\left(x + 1\right)^2 - 3 + 7 \\ &= 3\left(x + 1\right)^2 + 4 \end{aligned}$$

Worked Example 4

(a) Complete the square for $y = 2x^2 - 8x + 2$.

(b) Find the minimum value of y.

(c) Sketch the graph of $y = 2x^2 - 8x + 2$.

Solution

(a) First rearrange the quadratic as shown.

$$2x^2 - 8x + 2 = 2\left(x^2 - 4x\right) + 2.$$

Then $x^2 - 4x$ can be written as $\left(x - 2\right)^2 - 4$ to give

$$\begin{aligned} 2\left(x^2 - 4x\right) + 2 &= 2\left[\left(x - 2\right)^2 - 4\right] + 2 \\ &= 2\left(x - 2\right)^2 - 8 + 2 \\ &= 2\left(x - 2\right)^2 - 6 \end{aligned}$$

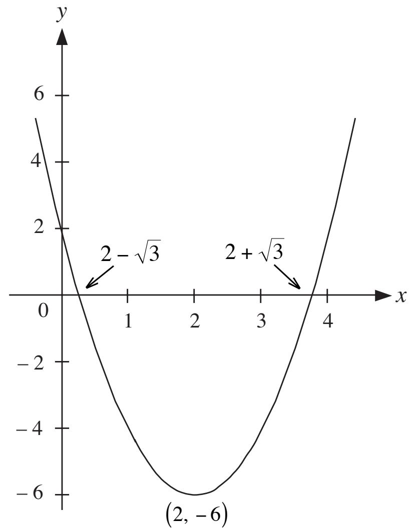
(b) As $y = 2\left(x - 2\right)^2 - 6$, the minimum possible value of y is -6, which is obtained when $x - 2 = 0$ or $x = 2$.

(c) Before sketching the graph, it is also useful to find where the curve crosses the x-axis, that is when $y = 0$. To do this, solve

$$\begin{aligned} 0 &= 2\left(x - 2\right)^2 - 6 \\ 2\left(x - 2\right)^2 &= 6 \\ \left(x - 2\right)^2 &= 3 \\ x - 2 &= \pm\sqrt{3} \\ x &= 2 \pm \sqrt{3} \end{aligned}$$

So the curve crosses the x-axis at $2+\sqrt{3}$ and $2-\sqrt{3}$, and has a minimum at $(2, -6)$.

This is shown in the graph opposite.



Worked Example 5

(a) Express $3x^2+2x+1$ in the form $a(x+p)^2+q$ where a, p and q are real numbers.

(b) Hence, determine for $f(x)=3x^2+2x+1$

(i) the minimum value for $f(x)$

(ii) the equation of the axis of symmetry.

Solution

(a)
$$\begin{aligned} 3x^2+2x+1 &= a(x+p)^2+q \\ &= a\left(x^2+2px+p^2\right)+q \\ &= ax^2+2apx+\left(ap^2+q\right) \end{aligned}$$

Equating coefficients:

$$\left[x^2\right] \qquad 3 = a \quad \Rightarrow a = 3$$

$$[x] \qquad 2 = 2ap \Rightarrow p = \frac{2}{2a} = \frac{1}{3}$$

$$\left[c^t\right] \qquad 1 = ap^2+q = 3\times\left(\frac{1}{3}\right)^2+q$$

$$= \frac{1}{3}+q$$

$$q = 1-\frac{1}{3} = \frac{2}{3}$$

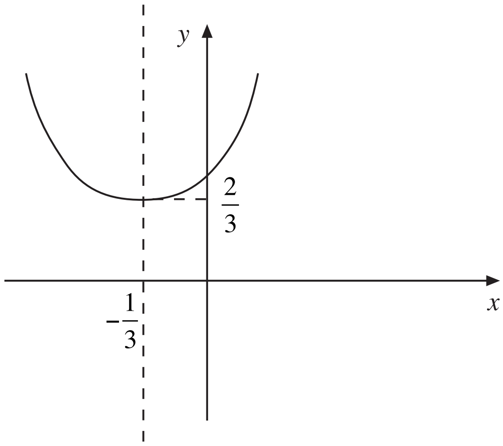
Thus

$$3x^2+2x+1 = 3\left(x+\frac{1}{3}\right)^2+\frac{2}{3}$$

(b) (i) Minimum value of $y = 3x^2 + 2x + 1$ will occur when $x + \frac{1}{3} = 0$; that is,

$x = -\frac{1}{3}$, and the value is $y = \frac{2}{3}$.

(ii)



$x = -\frac{1}{3}$ is the equation of the axis of symmetry.

Exercises

1. Complete the square for each of the expressions below.

(a)	$x^2 + 4x - 5$	(b)	$x^2 + 6x - 1$	(c)	$x^2 + 10x - 2$
(d)	$x^2 - 8x + 2$	(e)	$x^2 + 12x + 3$	(f)	$x^2 - 20x + 10$
(g)	$x^2 + 3x - 1$	(h)	$x^2 - 5x + 2$	(i)	$x^2 - x + 4$

2. Use the completing the square method to solve each of the following equations.

(a)	$x^2 - 4x + 3 = 0$	(b)	$x^2 - 6x - 4 = 0$	(c)	$x^2 + 10x - 8 = 0$
(d)	$x^2 + 5x + 1 = 0$	(e)	$x^2 + x - 1 = 0$	(f)	$x^2 + 2x - 4 = 0$
(g)	$x^2 + 4x - 8 = 0$	(h)	$x^2 + 5x - 2 = 0$	(i)	$x^2 + 7x + 1 = 0$

3. Complete the square for each of the following expressions.

(a)	$2x^2 + 8x - 1$	(b)	$2x^2 + 10x - 3$	(c)	$2x^2 + 2x + 1$
(d)	$3x^2 + 6x - 2$	(e)	$5x^2 + 15x - 4$	(f)	$7x^2 - 14x + 2$
(g)	$3x^2 + 12x - 4$	(h)	$4x^2 + 20x - 3$	(i)	$2x^2 - 12x + 3$

4. Solve each of the following equations by completing the square.

(a) $2x^2 + 4x - 5 = 0$ (b) $2x^2 + 16x - 3 = 0$

(c) $3x^2 + 12x - 8 = 0$ (d) $4x^2 + 2x - 1 = 0$

(e) $2x^2 + x - 6 = 0$ (f) $5x^2 - 20x + 1 = 0$

5. Sketch the graph of each equation below, showing its minimum or maximum point and where it crosses the x-axis.

(a) $y = x^2 - 2x - 1$ (b) $y = x^2 + 6x + 8$

(c) $y = x^2 - 10x + 24$ (d) $y = x^2 + 5x - 14$

(e) $y = 4 + 3x - x^2$ (f) $y = 3x - 2 - x^2$

6. The height of a ball thrown into the air is given by

$$h = 1 + 20t - 10t^2$$

Find the maximum height reached by the ball.

7. (a) By writing the quadratic expression

$$x^2 - 4x + 2$$

in the form $(x + a)^2 + b$, find a and b and hence find the minimum value of the expression.

(b) Solve the equation

$$x^2 - 4x + 2 = 0$$

giving your answers correct to 2 decimal places.

8. (a) Factorise the expression $x^2 + 2x - 3$.

(b) Express $x^2 + 2x - 3$ in the form $(x + a)^2 - b$, where a and b are whole numbers.

(c) Sketch the curve with equation $y = x^2 + 2x - 3$.

9. (a) Express the function $f(x) = 2x^2 - 4x - 13$ in the form $f(x) = a(x + h)^2 + k$.

Hence, or otherwise, determine

(b) the values of x at which the graph cuts the x-axis.

(c) the interval for which $f(x) \le 0$

(d) the minimum value of $f(x)$

(e) the value of x at which $f(x)$ is a minimum.

10. (a) If $4y^2 + 3y + b$ is a perfect square, calculate the value of b.

(b) By the method of completing the square, solve the equation $5y^2 = 8y - 2$. Give your answers to 3 significant figures.

Information

The word 'quadratic' comes from the Latin word 'quadratum', which means 'a squared figure'.